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Oscillating Universe and Scalar Field

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Abstract

In order to obtain models of the homogeneous isotropic universe that can oscillate without going through a singular state, a cosmic field is introduced that produces a negative pressure, following the work of Pachner. One is led to single out a particular form for this field. If one adds to the Einstein field equations an expression corresponding to this field, taking into account the existence of a cosmic time, one obtains the C-field of Hoyle and Narlikar for the case of conservation of matter.

1. *Introduction*

It is now generally believed that the homogeneous isotropic expanding model of the universe, in the framework of the general theory of relativity, represents a fairly good approximation to the real universe. From the aesthetic point of view, the most attractive model is one which is spatially closed and oscillates periodically in time. For if the universe is closed, then no question of boundary conditions arises, and if it is periodic in time, there is no question of initial conditions. Of course, we have no assurance that nature behaves in accordance with our views, and ultimately the model chosen will be the one that gives the best agreement with observational data. However, at the present time the accuracy of the data seems to be inadequate to single out a particular model. It seems desirable therefore to hold on to the closed periodic model of the universe as long as there is no observational evidence against it.

The simplest model which is spatially closed is one having positive curvature. The sign of the curvature can be taken arbitrarily in the line element of general relativity and there does not appear to be any objection in principle to taking it positive. To be sure, the question can be raised whether the sign of the curvature is determined by more fundamental considerations (such as those based on Mach's principle, for example) but that need not concern us here.

On the other hand, the oscillatory behaviour of the model must come out of the equations of motion for the radius of the universe as a function of

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the time. Here difficulties arise. If one considers a universe containing matter and radiation, characterised by density and pressure, one finds (Tolman, 1934) that the only possible oscillations are those for which the minimum value of the radius is zero, corresponding to an infinite density. To avoid such a singular state, Pachner (1965) assumed the existence of a negative pressure. Following McCrea (1951), he considered this stress as arising from the properties of the vacuum. Taking the stress as proportional to R^{-n}, he investigated the case $n=4$ and showed that oscillations were possible for which the radius was finite and there was therefore no singular state. The work of Pachner represents the starting point of the present discussion.

2. *Cosmic Field*

The line element of the homogeneous isotropic universe can be written

$$ds^2 = -\frac{f^2}{\left(1+\dfrac{kr^2}{4R_0^2}\right)^2}(dx^2+dy^2+dz^2)+dt^2 \tag{2.1}$$

where $f=f(t)>0$ is the scale factor, R_0 is a constant having dimensions of length, $k=\pm 1$ or 0, characterises the nature of the spatial curvature, and $r^2=x^2+y^2+z^2$. If we define the radius of the universe as $R=R(t)$, given by

$$R=R_0 f, \tag{2.2}$$

then the Einstein field equations reduce to the following relations:

$$\frac{2\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}=-8\pi p-\frac{k}{R^2}+\Lambda \tag{2.3}$$

$$\frac{\dot{R}^2}{R^2}=\frac{8\pi}{3}\rho-\frac{k}{R^2}+\frac{\Lambda}{3} \tag{2.4}$$

Here Λ is the cosmological constant, ρ and p are the density and pressure, respectively, of the matter and radiation, and a dot denotes differentiation with respect to t. From equations (2.3) and (2.4) one obtains

$$\frac{d}{dR}(\rho R^3)+3pR^2=0 \tag{2.5}$$

If ρ and p satisfy equation (2.5), then it is sufficient to consider equation (2.4) as determining $R(t)$, and one need not make use of equation (2.3).

Following Pachner (1965), let us now assume that space, as well as matter, can be characterised by quantities ρ and p which satisfy equation (2.5). Such an assumption no longer seems strange, now that we have learned from quantum field theory about electromagnetic field fluctuations and virtual particle pairs in the vacuum. It is natural to suppose that ρ and p are proportional,

$$p=\alpha\rho \qquad (\alpha=\text{const.}) \tag{2.6}$$

Substituting this relation into equation (2.5) one finds that

$$\rho = AR^{-n} \tag{2.7}$$

where A is an arbitrary constant and

$$n = 3(1 + \alpha) \tag{2.8}$$

In Table 1 are given some values of α in the range $(-1,+1)$ and the corresponding values of n. In the right-hand column there is a description of some of the corresponding fields. One sees several familiar cases included,

TABLE 1

α	n	Description of field
-1	0	Cosmological constant
$-\frac{2}{3}$	1	
$-\frac{1}{3}$	2	Curvature
0	3	Dust
$\frac{1}{3}$	4	Radiation
$\frac{2}{3}$	5	
1	6	Cosmic field

such as the density of dust ($n = 3$) and of radiation ($n = 4$). One can consider the cosmological constant appearing in equation (2.4) as a kind of field, corresponding to $n = 0$, as was pointed out by Pachner. We also see that for $n = 2$ we get what might be called a curvature field, corresponding to the second term in the right-hand member of equation (2.4).

However, the most interesting case in Table 1 is perhaps that for $\alpha = 1$, $n = 6$. This is the case for which $dp/d\rho = 1$. If we had a material medium for which this relation held, acoustic waves would be transmitted through it with the speed of light. Hence this represents a limiting case and appears to have special significance. Let us write this

$$\rho_c = p_c = -\frac{3C^2}{8\pi R^6} \qquad (C = \text{const.}) \tag{2.9}$$

and let us refer to it as the cosmic field. The negative sign has been introduced in equation (2.9) in order to get the negative pressure that Pachner found to be needed in order to avoid the singular state. It appears therefore that ρ_c should not be regarded as an actual mass density. Rather, like the curvature term and the cosmological constant, it should be considered simply as a term appearing in the equation of motion of R, equation (2.4), it being associated with the properties of space. Later we shall see how the corresponding field appears in the Einstein field equations.

Taking ρ_c into account, we can now write in place of equation (2.4)

$$\frac{\dot{R}^2}{R^2} = -\frac{C^2}{R^6} + \frac{8\pi\rho}{3} - \frac{k}{R^2} + \frac{\Lambda}{3} \tag{2.10}$$

If we consider ρ as describing the density of matter and radiation, so that as $R \to 0$, $\rho \sim R^{-4}$, and as $R \to \infty$, $\rho \sim R^{-3}$, then we see that this equation permits oscillations without a singular state. For if we suppose that there is a domain of values of R for which ρ is sufficiently large to make the right-hand member of equation (2.10) positive, then by going to smaller values of R one must reach a finite value R_1 for which this member vanishes. This value R_1, will be the minimum value of R, corresponding to a non-singular state. If we assume that $k = 1$, $\Lambda = 0$, then by going to larger values of R we must again reach a value, R_2, for which this member vanishes, and this is the maximum value of R. From the form of equation (2.10), it follows that R will oscillate periodically between R_1 and R_2.

We see that for $k = 1$, corresponding to a closed universe, the cosmological constant is not required for oscillations to take place, and for the sake of simplicity one would like to omit it. However, if we regard the Λ-term as corresponding to a kind of vacuum field, according to Table 1, it is possible that this term is needed for a satisfactory description of nature. For $\Lambda < 0$, one obtains oscillations even if $k = 0$ or $k = -1$. However, if $\Lambda > 0$, then oscillations will occur only for $k = 1$, and then only provided Λ is not too large.

As to the order of magnitude of the first term on the right-hand side of equation (2.10), one can say that in the distant past, when the radius of the universe was near its minimum value, this term was large and comparable to the density term. At the present time, when the radius is very much larger, one can expect this term to be relatively unimportant.

3. *Examples*

Let us consider first a very simple, if unrealistic, example. This is the case in which $k = 0$, $\Lambda < 0$, and the matter consists of dust, so that one can write

$$\frac{8\pi\rho}{3} = \frac{2A^2}{R^3} \qquad (A = \text{const.}) \tag{3.1}$$

If we also write $\Lambda = -\frac{1}{3}\omega^2$, equation (2.10) takes the form

$$\frac{\dot{R}^2}{R^2} = -\frac{C^2}{R^6} + \frac{2A^2}{R^3} - \tfrac{1}{9}\omega^2 \tag{3.2}$$

This can be easily solved by taking as a new variable $y = R^3$. If the condition

$$A^4 > \tfrac{1}{9}\omega^2 C^2$$

is satisfied, then the solution can be written

$$R^3 = \frac{9A^2}{\omega^2} + \left(\frac{81A^4}{\omega^4} - \frac{9C^2}{\omega^2}\right)^{1/2} \sin(\omega t + \delta) \qquad (\delta = \text{const.}) \tag{3.3}$$

and this describes non-singular oscillations.

Now let us consider the case in which $k = 1$, $\Lambda = 0$, and the universe contains only radiation, so that

$$\frac{8\pi\rho}{3} = \frac{2B}{R^4} \qquad (B = \text{const.} > 0) \tag{3.4}$$

Equation (2.10) now takes the form

$$\frac{\dot{R}^2}{R^2} = -\frac{C^2}{R^6} + \frac{2B}{R^4} - \frac{1}{R^2} \tag{3.5}$$

Let us assume that

$$B > |C|$$

If one now takes as a new variable $z = R^2 - B$, the solution can be written

$$\int \left(\frac{z + B}{b^2 - z^2}\right)^{1/2} dz = 2t + \text{const.} \tag{3.6}$$

where

$$b^2 = B^2 - C^2$$

Let us now write

$$z = b \sin\left(2\theta + \frac{\pi}{2}\right) \tag{3.7}$$

Then equation (3.6) goes over into

$$(B + b)^{1/2} \int_0^\theta (1 - k^2 \sin^2\theta)^{1/2}\, d\theta = t + \text{const.} \tag{3.8}$$

with

$$k^2 = \frac{2b}{B + b} < 1$$

so that one has

$$E(k, \theta) = (B + b)^{-1/2}(t + \gamma) \qquad (\gamma = \text{const.}) \tag{3.9}$$

where $E(k, \theta)$ is one of the standard elliptic integrals (Jahnke *et al.*, 1960).

We see from equation (3.5) or (3.7) that the minimum and maximum values of R are

$$R_1 = (B - b)^{1/2}, \qquad R_2 = (B + b)^{1/2} \tag{3.10}$$

For any choice of the function $\rho(R)$ that gives a reasonably good approximation to the mean density of matter and radiation in the universe, one can expect that the solution of equation (2.10) will require numerical integration.

4. *Field Equations and Scalar Field*

The Einstein field equations, including the cosmological term, have the form

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu} \tag{4.1}$$

where, for a fluid characterised by a proper density ρ, a pressure p and a velocity $u^\lambda = dx^\lambda/ds$,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - g^{\mu\nu}p \tag{4.2}$$

In the case of the homogeneous isotropic model of the universe, since the coordinate system used in equation (2.1) is co-moving, one has

$$u^4 = u_4 = 1, \qquad u^k = u_k = 0 \qquad (k = 1, 2, 3) \tag{4.3}$$

and the field equations reduce to equations (2.3) and (2.4), as previously noted.

The question now arises: how should the above equations be modified in order to include the cosmic field which was introduced for the purpose of obtaining an oscillatory behaviour for the universe?

We have seen that formally we can describe the cosmic field by means of the quantities ρ_c and p_c, with $p_c = \rho_c$. If these quantities referred to a material medium having a velocity u^λ, we could substitute them into equation (4.2) to get a corresponding tensor $T_c^{\mu\nu}$, which could then be added to the energy-momentum density tensor of matter and radiation appearing on the right-hand side of equation (4.1). However, we are dealing here with the properties of space, and not of a material medium, and this presents us with a difficulty.

In the case of a homogeneous isotropic universe, there exists a preferred frame of reference, the co-moving coordinate system, with a preferred time, often called the cosmic time, given by a clock at rest in this reference frame. Let us refer to this as the fundamental system and let us denote it by S_0. Thus the line element of equation (2.1) refers to S_0. Now one can show (Rosen, 1969a, b) that an observer in a laboratory which is moving freely through space can, in principle, determine his motion relative to S_0 by means of mechanical or optical experiments performed inside the laboratory. Thus one has a somewhat paradoxical situation: although one is working in the framework of the general theory of relativity, one encounters concepts quite similar to those of absolute space, time and motion in classical physics, concepts which one would tend to regard as being in disagreement with the basic ideas of general relativity. This suggests that, if the homogeneous isotropic model represents a reasonable approximation to the real universe, then the foundations of the general relativity theory need further investigation and, perhaps, revision.

Let us go back to the velocity vector u^λ which, in S_0, satisfies equations (4.3). This is, of course, a time-like vector, and one readily verifies that in S_0, and hence in every coordinate system, it satisfies the following relations:

$$u_\lambda u^\lambda = 1 \tag{4.4}$$

$$u_{\lambda;\nu}u^\nu = 0 \tag{4.5}$$

and

$$u_{\lambda;\nu} - u_{\nu;\lambda} \equiv u_{\lambda,\nu} - u_{\nu,\lambda} = 0 \tag{4.6}$$

where a comma denotes an ordinary partial derivative and a semicolon a covariant derivative. These relations will all be satisfied if one writes

$$u_\lambda = \psi_{,\lambda} \tag{4.7}$$

where ψ is a scalar, such that

$$g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} = 1 \tag{4.8}$$

In S_0 one obviously has

$$\psi = t + \text{const.} \tag{4.9}$$

so that ψ can be regarded as describing the cosmic time.

Let us now put aside the definition of u^λ as the matter velocity vector, and let us consider u^λ simply as a vector determining a preferred direction in space-time. In a given coordinate system the components of this vector are to be obtained by a vector transformation from the components in S_0 given by equations (4.3). The assumption of the existence of such a vector appears to be at variance with the foundations of general relativity. Nevertheless it is justified in the case of the homogeneous isotropic universe because of the existence of a preferred system, the fundamental system S_0. One might think of this vector as pointing in the 'direction of flow' of the cosmic time.

If we write

$$\rho_c = p_c = -\frac{\phi^2}{8\pi} \tag{4.10}$$

where ϕ is a scalar then, corresponding to equation (4.2), we can take formally

$$8\pi T_c^{\mu\nu} = \phi^2(g^{\mu\nu} - 2u^\mu u^\nu) \tag{4.11}$$

which should be subtracted from the right-hand member of equation (4.1) to describe the effect of the cosmic field. This field equation can now be written in the form

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + \phi^2(g_{\mu\nu} - 2u_\mu u_\nu) = -8\pi T_{\mu\nu} \tag{4.12}$$

Since $T^{\mu\nu}$ has zero divergence, it follows that we must also have

$$[\phi^2(g^{\mu\nu} - 2u^\mu u^\nu)]_{;\nu} = 0 \tag{4.13}$$

This equation, which determines ϕ, can be written

$$\phi_{,\nu}(g^{\mu\nu} - 2u^\mu u^\nu) - \phi u^\mu u^\nu{}_{;\nu} = 0 \tag{4.14}$$

Multiplying by u_μ, one gets

$$(\phi u^\nu)_{;\nu} = 0 \tag{4.15}$$

so that equation (4.14) can be put into the form

$$\phi_{,\mu} = u_\mu \phi_{,\nu} u^\nu = -\phi u_\mu u^\nu{}_{;\nu} \tag{4.16}$$

In the co-moving system one sees that $u^{\nu}{}_{;\nu} = 3\dot{R}/R$ and that $\phi = \phi(t)$, so that equation (4.16) reduces to

$$\dot{\phi} = -\frac{3\dot{R}}{R}\phi \tag{4.17}$$

Hence one has

$$\phi R^3 = \text{const.} \tag{4.18}$$

and ρ_c, as given by equation (4.10), has the same form as that obtained previously.

Having found the solution for the scalar ϕ in S_0, we know that it will be the solution in every other system. However, up to this point we have been considering the case of the homogeneous isotropic universe, and the question arises whether one can generalise the discussion to the case in which the space is inhomogeneous as, for example, in the vicinity of a star.

One can expect that in the general case it will not be possible for the vector u^{λ} to satisfy all the conditions imposed by equations (4.4)–(4.6). Let us therefore assume only that u^{λ} is a normalized time-like vector, i.e., that it satisfies equation (4.4). One can now define a vector N^{μ} by the relation

$$N^{\mu} = \phi u^{\mu} \tag{4.19}$$

so that

$$N_{\alpha} N^{\alpha} = \phi^2 > 0 \tag{4.20}$$

and one can express $T_c^{\mu\nu}$ in terms of it,

$$8\pi T_c^{\mu\nu} = N_{\alpha} N^{\alpha} g^{\mu\nu} - 2N^{\mu} N^{\nu} \tag{4.21}$$

Setting $T_c{}^{\mu\nu}{}_{;\nu}$ equal to zero gives the equation

$$(N_{\alpha;\mu} - N_{\mu;\alpha}) N^{\alpha} = N_{\mu} N^{\alpha}{}_{;\alpha} \tag{4.22}$$

Multiplying by N^{μ}, one gets

$$N^{\alpha}{}_{;\alpha} = 0 \tag{4.23}$$

so that equation (4.22) becomes

$$(N_{\lambda;\mu} - N_{\mu;\lambda}) N^{\lambda} = 0 \tag{4.24}$$

What is required therefore is a vector N_{μ} satisfying equations (4.23) and (4.24).

One can satisfy equation (4.24) by setting

$$N_{\lambda;\mu} - N_{\mu;\lambda} = 0 \tag{4.25}$$

so that the solution is then given by

$$N_{\mu} = \chi_{,\mu} \tag{4.26}$$

where χ is a scalar which, by equation (4.23), satisfies the D'Alembert equation

$$\chi_{;\alpha\underline{\alpha}} = 0 \tag{4.27}$$

Here a line under an index indicates that it is to be raised by means of $g^{\mu\nu}$. One now has

$$8\pi T_{(c)\mu\nu} = \chi_{,\alpha}\chi_{,\underline{\alpha}}g_{\mu\nu} - 2\chi_{,\mu}\chi_{,\nu} \tag{4.28}$$

We see that the scalar χ is essentially a special case of the C-field used by Hoyle & Narlikar (1964) in their 'continuous creation' cosmology. Indeed, they showed that, in the special case in which matter is conserved, the C-field leads to the singularity-free behaviour described by equation (2.10). In the present discussion it is, of course, taken for granted that the matter is conserved.

One can expect that equation (4.24) will have more general solutions than those satisfying equations (4.25) and (4.26). However, it may be that the latter are sufficiently general for the present purpose. It is rather pleasing to have the cosmic field described by a scalar function χ, the gradient of which can be considered as defining the direction of flow of cosmic time.

Making use of equation (4.28) we can now write the field equations in the form

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + \chi_{,\alpha}\chi_{,\underline{\alpha}}g_{\mu\nu} - 2\chi_{,\mu}\chi_{,\nu} = -8\pi T_{\mu\nu} \tag{4.29}$$

where χ satisfies equation (4.27). Far from the inhomogeneity, where the picture given by the homogeneous isotropic model is valid, if we are working in the fundamental system S_0, the appropriate solution of equation (4.27) must have the form

$$\chi = \chi(t) = K\int^{t}\frac{dt}{R^3} \qquad (K = \text{const.} > 0) \tag{4.30}$$

The solution of equation (4.27) must therefore be chosen so as to go over to this form at a large distance from the inhomogeneity.

Having found χ, one can then write

$$\phi^2 = \chi_{,\alpha}\chi_{,\underline{\alpha}} \tag{4.31}$$

$$u_\lambda = \frac{1}{\phi}\chi_{,\lambda} \tag{4.32}$$

and

$$\rho_c = -\frac{1}{8\pi}\chi_{,\alpha}\chi_{,\underline{\alpha}} \tag{4.33}$$

In the homogeneous region, where χ is given by equation (4.30), in the system S_0, one obtains agreement with equation (2.9) provided one chooses K so that

$$K^2 = 3C^2 \tag{4.44}$$

In conclusion, it should be remarked that we have arrived at the scalar χ on the basis of cosmological considerations, regarding it as being associated with the properties of space-time. It is therefore natural to suppose that there is no direct interaction between this scalar field and matter, just as in the case of the cosmological term, for example. In this respect the χ field differs from the scalar field of Dicke (1962). It is possible, of course, that the present standpoint may have to be abandoned in the future.

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